

Oscillator Quantum Algebra and Deformed $su(2)$ Algebra

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Abstract

A difference operator realization of quantum deformed oscillator algebra $\mathcal{H}_q(1)$ with a Casimir operator freedom is introduced. We show that this $\mathcal{H}_q(1)$ have a non-linear mapping to the deformed quantum $su(2)$ which was introduced by Fujikawa et al. We also examine the cyclic representation obtained by this difference operator realization and the possibility to analyze a Bloch electron problem by $\mathcal{H}_q(1)$.

Quantum deformed algebra[1] was firstly introduced to study the inverse scattering problems and the integral systems, which have rich structures, Yang-Baxter equations[2]. They are going to be standard techniques of theoretical physics. Wiegmann and Zabrodin found that a system of the Bloch electron on a two dimensional square lattice [5][6][7] can be expressed in a linear combination of generators of the algebra $U_q(sl_2)$. This corresponds to the fact that $U_q(sl_2)$ gives a foundation to obtain an exact solution of the Bethe Ansatz equation.

Bidenharn[3] and Macfarlane[4] introduced q-deformed oscillator algebra $\mathcal{H}_q(1)$ to construct $U_q(sl_2)$ in the manner of Schwinger's construction of conventional $su(2)$. Recently a new method to construct a representation of $\mathcal{H}_q(1)$, which is manifestly free of negative norm, was proposed[12]. This $\mathcal{H}_q(1)$ enjoys Hopf structure[14]. This algebra was used to analyze the phase operator problem [9][10] of the photon with the notion of index [13]. Fujikawa et al[8] constructed a new deformation of $su(2)$ algebra with a "Schwinger term"

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from this non-negative norm representation of $\mathcal{H}_q(1)$ in the manner of Schwinger's construction of conventional $su(2)$. We use the "Schwinger term" to represent an extra term which deforms the q -deformed $su(2)$, though a better terminology for it may exist. It was confirmed that their algebra coincide with $U_q(sl_2)$ algebra by choosing specific values of the deformation parameter.

In this paper we introduce a difference operator realization of $\mathcal{H}_q(1)$ with an additional term to prove nonlinear mapping from $\mathcal{H}_q(1)$ to deformed $su(2)$ with a Schwinger term. We also study a relation of a Bloch electron problem with the difference operator realization of deformed oscillator algebra. We show that Hamiltonian can be expressed in terms of generators of q -deformed oscillator algebra and that the Hamiltonian acts on the functional space on which q -deformed oscillator algebra is realized.

We analyze q -oscillator algebra introduced by Hong Yan[11].

$$\begin{aligned} [a, a^\dagger] &= [N+1]_q - [N]_q \\ [N, a^\dagger] &= a^\dagger \\ [N, a] &= -a \\ \mathcal{C}_2 &= a^\dagger a - [N]_q \end{aligned} \tag{1}$$

The usual notation of $[X]_q = \frac{q^X - q^{-X}}{q - q^{-1}}$ for deformation parameter $q = \exp(2\pi i\theta)$ with $-1/2 < \theta < 1/2$ is used. Following difference operators, which act on the functional space $f(x)$, are shown to realize this q -oscillator algebra $\mathcal{H}_q(1)$.

$$\begin{aligned} a^\dagger f(x) &= \mu x f(x), \\ a f(x) &= \mu^{-1} \frac{q^\lambda f(qx) - q^{-\lambda} f(q^{-1}x)}{(q - q^{-1})x} + \mu^{-1} \frac{c}{x} f(x), \\ q^N f(x) &= q^\lambda f(qx), \\ q^{-N} f(x) &= q^{-\lambda} f(q^{-1}x), \end{aligned} \tag{2}$$

where λ , μ and c are undetermined parameters. In the case special of $\lambda = 0$, $\mu = 1$ and $c = 0$, this difference realization coincide that of Ref.[11]. In the realization (2) the Casimir operator \mathcal{C}_2 has a value,

$$\mathcal{C}_2 f(x) = (a^\dagger a - [N])f(x) = (aa^\dagger - [N+1])f(x) = cf(x). \tag{3}$$

We note that when q is a root of unity we have additional central elements of $\mathcal{H}_q(1)$ and their values depend on parameters λ and μ . Then we can understand that the defining

relations (2) of $\mathcal{H}_q(1)$ on the functional space contain the degrees of freedoms which will correspond to the values of the central elements.

On this functional space we can construct a Fock space with generic parameters λ , μ and c . The operator N and $a^\dagger a$ commute with each other, and thus they can be diagonalized simultaneously on the state $|\psi_0\rangle$ [15],

$$N|\psi_0\rangle = \nu_0|\psi_0\rangle, \quad a^\dagger a|\psi_0\rangle = \lambda_0|\psi_0\rangle. \quad (4)$$

Our representation of $\mathcal{H}_q(1)$ has a Casimir operator \mathcal{C}_2 , and we obtain an additional constraint $c = \lambda_0 - [\nu_0]_q$ by acting \mathcal{C}_2 on $|\psi_0\rangle$. We impose a highest weight state condition $\lambda_0 = 0$, and we can determine ν_0 by c ,

$$c = -[\nu_0]_q. \quad (5)$$

All the states are generated by applying a^\dagger on the highest weight state, $|\psi_n\rangle = a^{\dagger n}|\psi_0\rangle$. We have for $n \geq 0$,

$$N|\psi_n\rangle = (\nu_0 + n)|\psi_n\rangle, \quad a^\dagger|\psi_n\rangle = |\psi_{n+1}\rangle, \quad a|\psi_n\rangle = \lambda_n|\psi_{n-1}\rangle, \quad \lambda_n = [n + \nu_0]_q - [\nu_0]_q \quad (6)$$

Those Fock states $|\psi_n\rangle$ can be constructed explicitly on the functional space up to normalization,

$$|\psi_n\rangle \sim \mu^n x^{\nu_0 - \lambda + n}. \quad (7)$$

In fact, the equations (4) and (6) are easily checked by using $N = x\partial_x + \lambda$, which is derived from difference operator realization in (2). This Fock space certainly contains an ordinary representation of $\mathcal{H}_q(1)$. But if we want to construct a cyclic representation from a finite dimensional irreducible representation of $\mathcal{H}_q(1)$, which can be obtained for q being a root of unity, we ought to clear some subtle points. This is because in order to obtain a cyclic representation, the Fock space should have integral powers in x . We thus need to examine the functional space (7), whose bases do not always have integral powers in x .

We can have a finite dimensional irreducible Fock space, if a state is truncated as $aa^\dagger|\psi_k\rangle = a|\psi_{k+1}\rangle = \lambda_{k+1}|\psi_k\rangle = 0$. So, it is enough to examine a condition $\lambda_{k+1} = 0$. For the generic θ , we can have a finite dimensional irreducible representation by choosing a suitable value of ν_0 [15]. Let us consider the rational $\theta = P/Q$, where P and Q are

relatively prime. The condition $\lambda_{k+1} = 0$, which determines if the truncation of states occur or not, is

$$\cos\left(\pi\frac{P}{2Q}(2\nu_0 + k + 1)\right)\sin\left(\pi\frac{P}{2Q}(k + 1)\right) = 0. \quad (8)$$

Then we can have a finite irreducible representation if this condition is satisfied. The cyclic representation of the algebra (2) can be obtained with the condition (8), by setting $x = q^k$ in (2). The cyclic basis are defined by,

$$f_k = f(q^k), \quad f_{k+2Q} = f_k. \quad (9)$$

Only the functions which satisfy $f(q^{2Q}x) = f(x)$ are admissible to construct this cyclic basis. This means that the function $f(x)$ should consist of integral powers of x , that is,

$$\nu_0 - \lambda \in \mathbb{Z}, \quad (10)$$

in (7). If the above conditions (8) and (10) are satisfied we can have a cyclic representation ρ of $\mathcal{H}_q(1)$ by inserting $x = q^k$ in (2).

$$\begin{aligned} \rho(a^\dagger)f_k &= \mu q^k f_k, \\ \rho(a)f_k &= \mu^{-1}(q - q^{-1})^{-1}q^{-k} \left(q^\lambda f_{k+1} - q^{-\lambda} f_{k-1} \right) + \mu^{-1}c q^{-k} f_k, \\ \rho(q^N)f_k &= q^\lambda f_{k+1}, \\ \rho(q^{-N})f_k &= q^{-\lambda} f_{k-1}. \end{aligned} \quad (11)$$

Let \mathcal{A} be a cyclic representation of the algebra, then \mathcal{A} has following properties.

$$\rho(\alpha a + \beta b)\vec{f} = \alpha\rho(a)\vec{f} + \beta\rho(b)\vec{f}, \quad \rho(ab)\vec{f} = \rho(b)\rho(a)\vec{f}, \quad (12)$$

where $a, b \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$ and cyclic bases $\vec{f} = (f_1, \dots, f_{2Q})^t$. This cyclic representation of the generators of $\mathcal{H}_q(1)$ can be expressed by using Weyl bases X and Y .

$$\begin{aligned} \rho(a^\dagger)\vec{f} &= \mu Y \vec{f}, \\ \rho(a)\vec{f} &= \mu^{-1}(q - q^{-1})^{-1}Y^{-1} \left(q^\lambda X^{-1} - q^{-\lambda} X \right) \vec{f} + \mu^{-1}c Y^{-1} \vec{f}, \\ \rho(q^N)\vec{f} &= q^\lambda X^{-1} \vec{f}, \\ \rho(q^{-N})\vec{f} &= q^{-\lambda} X \vec{f}. \end{aligned} \quad (13)$$

We have used the following $2Q \times 2Q$ matrix realization[6] of Heisenberg-Weyl group,

$$X = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \cdots & & \\ & & & \cdots & 0 \\ & & & & 1 & 0 \end{bmatrix}, \quad Y = \text{diag}(q, \cdots, q^{2Q}) \quad (14)$$

In this realization X and Y satisfy $qXY = YX$ and $Y^{2Q} = X^{2Q} = 1$.

Next we will show that the difference operator realization (2) of $\mathcal{H}_q(1)$ can be mapped to a deformed $su(2)$ algebra with a Schwinger term. This deformed $su(2)$ algebra is defined by the following relations[8],

$$\begin{aligned} [S_3, S_{\pm}] &= \pm S_{\pm}, \\ [S_+, S_-] &= [2S_3]_q + \xi[S_3]_q. \end{aligned} \quad (15)$$

The last term in (15) which is proportional to the ξ , gives rise to an extra term in the conventional q-deformed $su(2)$ algebra. This extra term play an essential role to construct a positive norm representation, and it has been tentatively called ‘‘Schwinger term’’. If $\xi = 0$ holds, this algebra is the same as the conventional q-deformed $su(2)$. The $2j + 1$ dimensional highest weight representation of the algebra (15) can also be realized by q-difference equations as

$$\begin{aligned} S_+ f(x) &= (q - q^{-1})^{-1} x (q^{2j - n_0 - \kappa} f(q^{-1}x) - q^{-2j + n_0 + \kappa} f(qx)) + x[n_0]f(x), \\ S_- f(x) &= -(q - q^{-1})^{-1} x^{-1} (q^{n_0 - \kappa} f(q^{-1}x) - q^{-n_0 + \kappa} f(qx)) + x^{-1}[n_0]f(x), \\ q^{S_3} f(x) &= q^{\kappa - j} f(qx), \\ \xi &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q - q^{-1})[n_0]_q[n_0 - j - \frac{1}{2}]_q, \end{aligned} \quad (16)$$

where we introduced n_0 to parameterize ξ , and κ is an undetermined parameter. This representation satisfies the highest weight condition $\tilde{S}_+ x^{2j - \kappa} = 0$ and the lowest weight condition $\tilde{S}_- \cdot x^{-\kappa} = 0$. We have $\{x^{-\kappa}, x^{-\kappa+1}, \cdots, x^{-\kappa+2j}\}$ for bases of this $2j + 1$ dimensional representation.

The deformed $su(2)$ with a Schwinger term (15) is *non-linearly* realized by using the generators of q-deformed oscillator $\mathcal{H}_q(1)$. Comparing difference operators in (2) and (16),

we can understand that there exist following realizations of S_+ , S_- and S_3 in terms of the generators of $\mathcal{H}_q(1)$,

$$\begin{aligned} S_+ &= (-[N + \delta]_q + c)a^\dagger, \\ S_- &= a, \\ S_3 &= N + \gamma, \end{aligned} \tag{17}$$

where the overall factor μ in (2) can always be removed in defining relations (16) of deformed $su(2)$, and thus we can set $\mu = 1$. We also make the following identifications,

$$c = [n_0]_q, \quad q^\lambda = q^{-n_0+\kappa}, \quad q^\delta = q^{2n_0-2j-1+\kappa}, \quad q^\gamma = q^{n_0-j+\kappa}. \tag{18}$$

Inserting this choice (18) of parameters into (17) we have the difference operator realizations of q-deformed $su(2)$. Parameters c and λ of $\mathcal{H}_q(1)$, and also δ and γ introduced to construct the mapping are determined by the data of the $2j+1$ dimensional representation of deformed $su(2)$, namely n_0 , j and k . This mapping (18) is different from the original realization of the deformed $su(2)$ which was constructed from two kinds of q-deformed oscillators[8].

We next study if the difference operator realization of $\mathcal{H}_q(1)$ can be used to analyze a Bloch electron on the square lattice. A Hamiltonian with two anisotropic parameters V_1 and V_2 can be written on the cyclic basis (9) with momentum p_\pm by using X and Y [6],

$$H = V_1 e^{i(p_++p_-)} Y^{-1} X^{-1} + V_2 e^{i(p_+-p_-)} X^{-1} Y + V_1 e^{-i(p_++p_-)} X Y + V_2 e^{-i(p_+-p_-)} Y^{-1} X. \tag{19}$$

We examine if this Hamiltonian can be expressed in terms of the generators of $\mathcal{H}_q(1)$ which realize the cyclic representation (11). We can use a following ansatz,

$$\begin{aligned} H &= \epsilon_1 \rho(a) + \epsilon_2 \rho(q^N) \rho(a^\dagger) + \epsilon_3 \rho(q^{-N}) \rho(a^\dagger) \\ &= \epsilon_1 \mu^{-1} (q - q^{-1})^{-1} Y^{-1} (q^\lambda X^{-1} - q^{-\lambda} X) + \epsilon_1 \mu^{-1} c Y^{-1} + \epsilon_2 q^\lambda X^{-1} \mu Y + \epsilon_3 q^{-\lambda} X \mu Y. \end{aligned} \tag{20}$$

Comparing those two expressions of Hamiltonian (19) and (20), we have conditions for ϵ_1 , ϵ_2 , ϵ_3 and c ,

$$\begin{aligned} \epsilon_1 \mu^{-1} (q - q^{-1})^{-1} q^\lambda &= V_1 e^{i(p_++p_-)}, \quad -\epsilon_1 \mu^{-1} (q - q^{-1})^{-1} q^{-\lambda} = V_2 e^{-i(p_+-p_-)}, \\ \epsilon_2 q^\lambda \mu &= V_2 e^{i(p_+-p_-)}, \quad \epsilon_3 q^{-\lambda} \mu = V_1 e^{-i(p_++p_-)}, \quad c = 0. \end{aligned} \tag{21}$$

In order to keep the generators of $\mathcal{H}_q(1)$ certainly corresponding to the cyclic representation, we also need the condition (10). We should check if $c = 0$ (21) and $\nu_0 - \lambda \in Z$ (10) conditions are satisfied or not. $c = 0$ means $q^{2\nu_0} = 1$ from (5), and $\nu_0 - \lambda \in Z$ means $q^{2\lambda} = q^{2\nu_0+2\kappa} = q^{2\kappa}$ with $\kappa \in Z$. Then we have the condition,

$$\left(-\frac{V_1}{V_2}\right) e^{2ip_+} = q^{2\kappa}. \quad (22)$$

This condition (22) leads to an isotropic and a midband condition, $V_1 = V_2$, and $p_+ = \pi/2 \bmod \pi/Q$. The first two equations in (21) can be satisfied simultaneously by $-e^{2ip_+} = q^{2\lambda}$. Then we can determine the parameter λ as $q^\lambda = \pm i e^{ip_+}$. Let us insert this solutions for ϵ_1, ϵ_2 and ϵ_3 into the Hamiltonian (20),

$$H = i(q - q^{-1}) \left(\rho(a) + \rho([N]_q) \rho(a^\dagger) \right). \quad (23)$$

where we set $e^{ip_-} \mu = 1$ and $V_1 = 1$ for simplicity and choose plus sign in (23) which comes from the condition $q^\lambda = \pm i e^{ip_+}$. By considering eq (8), we can find that the Fock space is truncated at $k = Q - 1$ for $P = \text{odd}$ and even, and the Fock space becomes the Q dimensional representation space $|\psi_i\rangle, i = 0, 1, \dots, k = Q - 1$. This means that Hamiltonian which is realized by $\mathcal{H}_q(1)$ on the functional space with integral powers certainly corresponds to a cyclic representation by inserting $x = q^k$ as usual way.

Because we constructed the relations (17) between $\mathcal{H}_q(1)$ and deformed $su(2)$ with a Schwinger term, the Hamiltonian (23) can be rewritten in terms of deformed $su(2)$, $H = -(q - q^{-1})(S_- - S_+)$ with a suitable choice of κ , say $\kappa = 0$. We have constraints $[n_0] = c = 0$ by (18) and (21), then the Schwinger term in deformed $su(2)$ vanishes. Then S_+, S_- and S_3 form the conventional q -deformed $su(2)$.

We note that since the deformation parameter $q = \exp(i\pi P/Q)$ satisfies $q^{2Q} = 1$, we can rewrite the Hamiltonian (23) in another form,

$$H = i(q - q^{-1}) \left(\rho(a) + q^Q \rho([N + Q]_q) \rho(a^\dagger) \right). \quad (24)$$

Using this alternative expression of the Hamiltonian (24), we can introduce generators of $U_q(sl_2)$, by $B = -[N + Q]_q a^\dagger, C = a, A = q^{N + \frac{1+Q}{2}}$ and $D = q^{-N - \frac{1+Q}{2}}$. The Hamiltonian can be written in this identification $H = i(q - q^{-1})(C \pm B)$, $P = \text{odd, even}$, respectively. This expression of Hamiltonian has been found by Wiegmann and Zabrodin to obtain the exact results of Bethe Ansatz equation[5][6].

In conclusion, we introduced a difference operator realization of $\mathcal{H}_q(1)$ with parameters which correspond to the degrees of freedom of central elements in the case of q being a root of unity and the value of the Casimir \mathcal{C}_2 . This realization of $\mathcal{H}_q(1)$ gives *non-linear* mapping from $\mathcal{H}_q(1)$ to the deformed $su(2)$ with a Schwinger term which was introduced by Fujikawa et al[8]. In the case of Schwinger term vanishing, we can deduce a correspondence between $\mathcal{H}_q(1)$ and the conventional q -deformed $su(2)$. We also examined if this difference realization of $\mathcal{H}_q(1)$ can describe the Hamiltonian of a Bloch electron problem on the square lattice. We found that the Hamiltonian has an expression in terms of generators of $\mathcal{H}_q(1)$ acting on the functional space. Using non-linear mapping from $\mathcal{H}_q(1)$ to the deformed $su(2)$, we can find that the Hamiltonian (24) coincide with what was found by Wiegmann and Zabrodin. We hope that the q -deformed oscillator algebra which realizes the Hamiltonian (23) gives further information about the Bloch electron problem in the future, since new mathematical machinery such as the coherent representation, etc., may become available.

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